

ON THE POISSON STRUCTURES RELATED TO κ -POINCARÉ GROUP.

PIOTR STACHURA

ABSTRACT. It is shown that the Poisson structure related to κ -Poincaré group is dual to certain Lie algebroid structure, the related Poisson structure on the (affine) Minkowski space is described in a geometric way.

1. INTRODUCTION

It is, more or less, “common knowledge” that the quantum κ -Poincaré Group [1] exists on a C^* -level, it is given by some bicrossproduct construction [2], [3] and it’s a quantization of certain Poisson-Lie structure [4]. Despite these beliefs, no precise and explicit formulae (e.g. for coproduct of generators) are known to the author. This note is a by-product of work on the C^* -version of the κ -Poincaré. It consists of two parts. In the first one, it is shown that really the Poisson structure presented in [4] is dual to certain Lie algebroid structure; this Lie algebroid is the Lie algebroid of a groupoid. The C^* -algebra of this groupoid should be the C^* -algebra of the quantum κ -Poincaré Group (it turns out we are in situation described in [5]). I tried to underline geometric and structural aspects of the construction. Such a formulation is necessary to study whether and in what sense κ -Poincaré group is a quantization of the Poincaré Group. The second part describes the Poisson version of the “ κ -Minkowski” (affine) space and its relation to the Poisson structure on the Poincaré Group. These are again rather simple observations (essentially this part is almost contained in [7]). In the second part, too, I tried to clarify geometric picture and present results in a coordinate-free form.

Notation for orthogonal Lie algebras. (V, η) stands for a real, finite dimensional vector space with a bilinear, symmetric and nondegenerate form η . An *orthonormal basis* is a basis (v_α) in V such that $\eta(v_\alpha, v_\beta) = \eta(v_\alpha, v_\alpha)\delta_{\alpha\beta}$, $|\eta(v_\alpha, v_\alpha)| = 1$. For a vector v with $|\eta(v, v)| = 1$ we write $\text{sgn}(v)$ for $\eta(v, v)$. By η we denote also the isomorphism $V \rightarrow V^*$ given by $\langle \eta(x), y \rangle := \eta(x, y)$. Using this notation, for any orthonormal basis (v_α) and any $x, y \in V$:

$$(1) \quad I = \sum_{\alpha} \text{sgn}(v_\alpha) v_\alpha \otimes \eta(v_\alpha), \quad \eta(x, y) = \sum_{\alpha} \text{sgn}(v_\alpha) \eta(x, v_\alpha) \eta(v_\alpha, y)$$

A subspace generated by vectors v_1, \dots, v_k is denoted by $\langle v_1, \dots, v_k \rangle$ or $\text{span}\{v_1, \dots, v_k\}$; for a subset $S \subset V$ the symbol S^\perp denotes *the orthogonal complement* of S , if $S = \{v\}$ we write v^\perp instead of $\{v\}^\perp$; the symbol $S^0 \subset V^*$ stands for *the annihilator* of S .

For vectors $x, y \in V$ let $\Lambda_{xy} := x \otimes \eta(y) - y \otimes \eta(x)$; for a basis (v_α) in V we write $\Lambda_{\alpha\beta}$ instead of $\Lambda_{v_\alpha, v_\beta}$. Operators Λ_{xy} satisfy:

$$(2) \quad [\Lambda_{xy}, \Lambda_{zt}] = \eta(x, t)\Lambda_{yz} + \eta(y, z)\Lambda_{xt} - \eta(x, z)\Lambda_{yt} - \eta(y, t)\Lambda_{xz}$$

and $\text{so}(\eta) = \text{span}\{\Lambda_{xy} : x, y \in V\}$. If $W \subset V$ is a subspace then $\Lambda_W := \text{span}\{\Lambda_{xy} : x, y \in W\}$ is a Lie subalgebra of $\text{so}(\eta)$; for a *null vector* $f \in V$ and a subspace $W \subset V$ the subspace $\Lambda_{Wf} := \text{span}\{\Lambda_{wf} : w \in W\}$ is also a subalgebra; notice that for $g \in O(\eta)$ we have $\Lambda_{gx, gy} = g\Lambda_{xy}g^{-1} =: \text{ad}(g)(\Lambda_{xy})$

We will use a bilinear, nondegenerate form $k : \text{so}(\eta) \times \text{so}(\eta) \rightarrow \mathbb{R}$ defined by:

$$(3) \quad k(\Lambda_{xy}, \Lambda_{zt}) := \eta(x, t)\eta(y, z) - \eta(x, z)\eta(y, t)$$

It is easy to see that for $g \in O(\eta)$: $\text{ad}(g) \in O(k)$ i.e.

$$k(g\Lambda_{xy}g^{-1}, g\Lambda_{zt}g^{-1}) = k(\Lambda_{xy}, \Lambda_{zt}), \quad g \in O(\eta)$$

(of course k is proportional to the Killing form). By $\text{ad}^\#$ we denote the coadjoint representation of $O(\eta)$ on $so(\eta)^*$: $\text{ad}^\#(g) := \text{ad}(g^{-1})^*$. If k is the isomorphism $so(\eta) \rightarrow so(\eta)^*$ defined by the form k then

$$\text{ad}^\#(g)k(X) = k(\text{ad}(g)X), \quad X \in so(\eta)$$

Let us also define a bilinear form \tilde{k} on $so(\eta)^*$ by:

$$(4) \quad \tilde{k}(\varphi, \psi) := k(k^{-1}(\varphi), k^{-1}(\psi)), \quad \varphi, \psi \in so(\eta)^*$$

so $\tilde{k}(\varphi, \psi) = \langle \varphi, k^{-1}(\psi) \rangle$; again it is clear that if $g \in O(\eta)$ then $\text{ad}^\#(g) \in O(\tilde{k})$, and

$$\tilde{k}(\text{ad}^\#(g)k(X), k(Y)) = k(\text{ad}(g)X, Y), \quad X, Y \in so(\eta)$$

We will also need one-parameter groups; they are given by formulae:

$$(5) \quad \begin{aligned} \exp(\Lambda_{uf}) &= I + \Lambda_{uf} - \frac{\eta(u, u)}{2} f \otimes \eta(f), \quad \eta(u, f) = 0 = \eta(f, f) \\ \exp(\nu \Lambda_{st}) &= I - P_{st} + \cosh(\nu) P_{st} + \sinh(\nu) \Lambda_{st}, \quad \eta(s, s) = -1 = -\eta(t, t), \quad \eta(s, t) = 0, \quad \nu \in \mathbb{R} \\ \exp(\nu \Lambda_{xy}) &= I - P_{xy} + \cos(\nu) P_{xy} + \sin(\nu) \Lambda_{xy}, \quad \eta(x, x) = \eta(y, y) = \pm 1, \quad \eta(x, y) = 0, \quad \nu \in \mathbb{R} \end{aligned}$$

where P_{vw} denotes the orthogonal projection onto $\langle v, w \rangle$. If vectors v, w are orthogonal and $|\eta(v, v)| = |\eta(w, w)| = 1$, then:

$$(6) \quad P_{vw} = \text{sgn}(v)v \otimes \eta(v) + \text{sgn}(w)w \otimes \eta(w) = -\text{sgn}(v)\text{sgn}(w)\Lambda_{vw}^2$$

Poincaré Group. Let (V, η) be a vector Minkowski space (signature of η is $(+, -, \dots, -)$). For a vector $v \in V$ with $\eta(v, v) \neq 0$ let R_v denote the reflection across the hyperplane v^\perp , i.e. $R_v = I - \frac{2}{\eta(v, v)}v \otimes \eta(v)$. The full orthogonal group $O(\eta)$ has four connected components: $SO_0(\eta)$ – the connected component of identity; $R_t SO_0(\eta)$, $\eta(t, t) > 0$ – the component containing time reflection; $R_s SO_0(\eta)$, $\eta(s, s) < 0$ – the component containing space reflection and $SO_1(\eta) := R_t R_s SO_0(\eta)$, $\eta(t, t) > 0, \eta(s, s) < 0$ – the component reversing time and space orientation (but keeping the space-time orientation intact). In this paper the *Poincaré Group* $P(\eta)$ will mean the semidirect product $V \rtimes O(\eta)$ and the *restricted Poincaré Group* $P_0(\eta)$ is $V \rtimes SO_0(\eta)$. Elements (w, g) of $P(\eta)$ act on V by affine mappings: $(w, g)(v) := w + gv$ and the group law is just the composition of these mappings: $(w, g)(u, h) = (w + gu, gh)$. Since $P(\eta)$ depends only on dimension n of V it will be also denoted by $P(n)$; also $O(\eta)$ will be denoted by $O(1, n-1)$.

2. POISSON-POINCARÉ GROUP.

The particular Lie-Poisson structure on Poincaré Group we are interested in was defined in [4]; it is dual to a certain Lie algebroid structure. The construction is as follows.

Let (V, η) be a vector Minkowski space of dimension $n+2$, $n > 1$ and $G := SO_0(\eta)$. Our Poisson-Poincaré Group will be realized as a subgroup of the semidirect product $\mathfrak{g}^* \rtimes G$:

$$(7) \quad (\varphi, g)(\psi, h) := (\varphi + \text{ad}^\#(g)\psi, gh)$$

Notice that if $H \subset G$ is a subgroup with a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ then $\mathfrak{h}^0 \times H$ is a subgroup of $\mathfrak{g}^* \rtimes G$.

Let us choose a (spacelike) vector $\mathbf{s} \in V$ with $\eta(\mathbf{s}, \mathbf{s}) = -1$ and define a subalgebra:

$$(8) \quad \mathfrak{a} := \text{span}\{\Lambda_{xy}, x, y \in \mathbf{s}^\perp\} = \{Y \in so(\eta) : Y\mathbf{s} = 0\}$$

It is straightforward to see that: $\mathfrak{a}^\perp = \text{span}\{\Lambda_{x\mathbf{s}}, x \in \mathbf{s}^\perp\}$ and $k(\Lambda_{x\mathbf{s}}, \Lambda_{y\mathbf{s}}) = \eta(x, y)$, $x, y \in \mathbf{s}^\perp$ i.e. (\mathfrak{a}^\perp, k) is an $n+1$ dimensional vector Minkowski space and the same is true for $(\mathfrak{a}^0, \tilde{k})$.

Let \tilde{A} be the connected subgroup of G with Lie algebra \mathfrak{a} : $\tilde{A} = \{g \in G : g\mathbf{s} = \mathbf{s}\} \simeq SO_0(1, n)$ (i.e. \tilde{A} is the proper, orthochronous Lorentz group). Therefore the subgroup $\mathfrak{a}^0 \times \tilde{A}$ is $P_0(n+1)$; this way we have identified $P_0(n+1)$ as a subgroup of the semidirect product $\mathfrak{g}^* \rtimes G$. For reasons which are related to the “quantum version” of our Poisson-Poincaré group, we will also consider the normalizer of \tilde{A} in G which will be denoted by A . It is easy to see that

$$(9) \quad A := \{g \in G : g\mathbf{s} = d(g)\mathbf{s}, d(g) = \pm 1\} = \tilde{A} \cup \exp(\pi \Lambda_{u\mathbf{s}})\tilde{A} = \tilde{A} \cup \tilde{A} \exp(\pi \Lambda_{us})$$

for any spacelike, normalized vector $u \in \mathfrak{s}^\perp$.

Let us now compute the action of $\exp(\pi\Lambda_{us})$ on \mathfrak{a}^0 :

$$\text{ad}^\#(\exp(\pi\Lambda_{us}))k(\Lambda_{xs}) = k(\exp(\pi\Lambda_{us})\Lambda_{xs}\exp(-\pi\Lambda_{us})), \text{ and}$$

$$\exp(\pi\Lambda_{us})\Lambda_{xs}\exp(-\pi\Lambda_{us}) = -\Lambda_{xs} - 2\eta(x, u)\Lambda_{us}, \quad \eta(u, u) = -1, \quad x, u \in \mathfrak{s}^\perp$$

In this way what exactly is $\mathfrak{a}^0 \times A$ depends on the dimension of V : for $n+1$ – even this is $P_0(n+1)$ extended by time reflection; for $n+1$ – odd this is $P_0(n+1)$ extended by space and time reflection.

The Lie-Poisson structure on $\mathfrak{a}^0 \times A$ depends on a choice of a timelike vector $\mathbf{t} \in V$ or, equivalently, on a splitting of \mathfrak{a}^0 into *space*: $\text{span}\{k(\Lambda_{us}), u \in \mathfrak{s}, \mathbf{t} \in \mathfrak{s}^\perp\}$ and *time*: $\langle k(\Lambda_{\mathbf{ts}}) \rangle$. So let us choose a (timelike) vector $\mathbf{t} \in \mathfrak{s}^\perp$, $\eta(\mathbf{t}, \mathbf{t}) = 1$; denote $\mathbf{f} := \mathbf{t} - \mathbf{s}$ and let us define subalgebras:

$$(10) \quad \begin{aligned} \mathfrak{c} &:= \text{span}\{\Lambda_{x\mathbf{f}} : x \in \mathfrak{s}^\perp\} = \text{span}\{\Lambda_{y\mathbf{f}} : y \in \mathfrak{t}^\perp\} \\ \mathfrak{b} &:= \text{span}\{\Lambda_{xy}, x, y \in \mathfrak{t}^\perp\} = \{Y \in \mathfrak{so}(\eta) : Y\mathbf{t} = 0\} \end{aligned}$$

The Lie algebra $\mathfrak{so}(\eta)$ can be decomposed as (direct sums of vector spaces):

$$(11) \quad \mathfrak{so}(\eta) = \mathfrak{c} \oplus \mathfrak{b} = \mathfrak{c} \oplus \mathfrak{a}$$

Let B, C be connected subgroups of G with Lie algebras $\mathfrak{b}, \mathfrak{c}$ respectively; then $B = \{g \in G : g\mathbf{t} = \mathbf{t}\} \simeq SO(n+1)$. Denote $U := \langle \mathfrak{s}, \mathbf{t} \rangle^\perp \subset V$; then $(U, -\eta)$ is an n dimensional (vector) Euclidean space. The subalgebra \mathfrak{c} can be decomposed further as

$$\mathfrak{c} = \Lambda_{U\mathbf{f}} \oplus \langle \Lambda_{\mathbf{ts}} \rangle,$$

where by (2) the first summand is an abelian ideal (in \mathfrak{c}).

Using (5) we obtain:

$$\exp(\nu\Lambda_{\mathbf{ts}})\exp(\Lambda_{u\mathbf{f}})\exp(-\nu\Lambda_{\mathbf{ts}}) = \exp(\Lambda_{(e^\nu u)\mathbf{f}}), \quad u \in U, \nu \in \mathbb{R}$$

Therefore $C = \{\exp(\Lambda_{u\mathbf{f}})\exp(\nu\Lambda_{\mathbf{ts}}) : u \in U, \nu \in \mathbb{R}\}$ and is isomorphic to the semidirect product $U \rtimes \mathbb{R}$ with group operation:

$$(12) \quad (u, \mu)(v, \nu) := (u + e^\mu v, \mu + \nu), \quad u, v \in U, \mu, \nu \in \mathbb{R}$$

The group C is the AN group in the Iwasawa decomposition $SO_0(1, n+1) = SO(n+1)(AN)$ i.e there is the equality $G = BC$.

The open set $\Gamma := AC \cap CA$ carries two differential groupoid structures over A and C [8]. The groupoid “responsible” for our Lie-Poisson structure is the groupoid $\Gamma_A : \Gamma \rightrightarrows A$. Namely, the bundle $(TA)^0 \subset T^*G$ is dual to the Lie algebroid $\mathcal{L}(\Gamma_A)$, which we realize as a vectors tangent in points of A to *left fibers* with bracket coming from *left invariant vector fields*. In this way $(TA)^0$ carries the canonical Poisson structure; on the other hand via right translation we can identify $(TA)^0$ with $\mathfrak{a}^0 \times A$ i.e. with the Poincaré group; it turns out this is a Poisson structure described in [4]. Let us compute Poisson brackets explicitly.

Algebroid structure. The map $A \times \mathfrak{c} \ni (a, p) \mapsto ap \in T_a\Gamma_A$ is a global trivialization of $\mathcal{L}(\Gamma_A)$. For $p \in \mathfrak{c}$ let X_p^L be the left invariant vector field on Γ_A defined by:

$$(13) \quad X_p^L(a) := ap$$

These vector fields satisfy:

$$(14) \quad [X_p^L, X_q^L] = X_{[p, q]}^L, \quad p, q \in \mathfrak{c}$$

and the anchor map $\Pi^L : \mathcal{L}(\Gamma_A) \rightarrow TA$ is given by

$$(15) \quad \Pi^L(X_p^L)(a) = P_{\mathbf{a}}(\text{ad}(a)p)a,$$

where $P_{\mathbf{a}}$ is the projection onto \mathfrak{a} corresponding to the decomposition $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{a}$. Short computations give:

$$(16) \quad P_{\mathbf{a}}\text{ad}(a)\Lambda_{x\mathbf{f}} = \text{ad}(a)\Lambda_{x\mathbf{t}} - d(a)\Lambda_{(ax)\mathbf{t}}, \quad x \in \mathfrak{s}^\perp, a \in A, as =: d(a)\mathbf{s}$$

The Poisson structure. Sections of $\mathcal{L}(\Gamma_A)$ define linear functions on $(TA)^0$, if X is a section of $\mathcal{L}(\Gamma_A)$, the corresponding function will be denoted by \tilde{X} . Explicit form of this function for X_p^L is:

$$\widetilde{X_p^L}(\varphi, a) = \langle \varphi a, X_p^L(a) \rangle = \langle \varphi a, ap \rangle = \langle \varphi, \text{ad}(a)p \rangle = \tilde{k}(\varphi, \text{ad}^\#(a)k(p)) , \varphi \in \mathfrak{a}^0 , p \in \mathfrak{c}$$

(in this formula $(TA)^0 \simeq \mathfrak{a}^0 \times A$ via right translations). The Poisson structure on $(TA)^0$ is defined by the brackets:

$$(17) \quad \{\widetilde{X_1}, \widetilde{X_2}\} = [\widetilde{X_1}, \widetilde{X_2}] , \{\tilde{X}, \pi^*(f_1)\} = \pi^*(\Pi^L(X)f_1) , \{\pi^*(f_1), \pi^*(f_2)\} = 0,$$

where f_1, f_2 are smooth functions on A , $\pi : T^*G \rightarrow G$ is the cotangent bundle projection and π^* denotes the pullback of functions.

Our Poincaré Group was identified with $\mathfrak{a}^0 \times A \simeq (TA)^0$ (via right translations). For $\varphi, \psi \in \mathfrak{a}^0$ let us define smooth functions $\tilde{k}_\varphi, \tilde{k}_{\varphi\psi}$ on $\mathfrak{a}^0 \times A$:

$$(18) \quad \begin{aligned} \tilde{k}_\varphi(\rho, a) &:= \tilde{k}(\varphi, \rho) \\ \tilde{k}_{\varphi\psi}(\rho, a) &:= \tilde{k}(\varphi, \text{ad}^\#(a)\psi) \end{aligned}$$

Any Poisson structure on $\mathfrak{a}^0 \times A$ is determined by brackets:

$$\{\tilde{k}_\varphi, \tilde{k}_\psi\} , \{\tilde{k}_\varphi, \tilde{k}_{\psi\rho}\} , \{\tilde{k}_{\varphi\lambda}, \tilde{k}_{\psi\rho}\} , \varphi, \psi, \rho, \lambda \in \mathfrak{a}^0$$

For the Poisson structure given by (17) we immediately get:

$$(19) \quad \{\tilde{k}_{\varphi\lambda}, \tilde{k}_{\psi\rho}\} = 0.$$

Let us now compute remaining brackets and compare them with the ones presented in [4]. To this end we will relate functions \tilde{k}_ψ and $\widetilde{X_p^L}$.

Lemma 2.1. *Let (ρ_α) be an orthonormal basis in \mathfrak{a}^0 and assume that elements $c_\alpha \in \mathfrak{c}$ satisfy $\tilde{k}(\psi, \text{ad}^\#(a)\rho_\alpha) = \langle \psi, \text{ad}(a)c_\alpha \rangle$ for any $\psi \in \mathfrak{a}^0$ and any $a \in A$. Then:*

$$\tilde{k}_\varphi = \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha} \widetilde{X_{c_\alpha}^L}$$

Proof: Indeed, using (1) let us compute:

$$\begin{aligned} \tilde{k}_\varphi(\psi, a) &= \tilde{k}(\varphi, \psi) = \tilde{k}(\text{ad}^\#(a^{-1})\varphi, \text{ad}^\#(a^{-1})\psi) = \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}(\text{ad}^\#(a^{-1})\varphi, \rho_\alpha) \tilde{k}(\rho_\alpha, \text{ad}^\#(a^{-1})\psi) = \\ &= \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}(\varphi, \text{ad}^\#(a)\rho_\alpha) \tilde{k}(\psi, \text{ad}^\#(a)\rho_\alpha) = \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha}(\psi, a) \tilde{k}(\psi, \text{ad}^\#(a)\rho_\alpha) = \\ &= \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha}(\psi, a) \langle \psi, \text{ad}(a)c_\alpha \rangle = \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha}(\psi, a) \widetilde{X_{c_\alpha}^L}(\psi, a) = \\ &= \left(\sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha} \widetilde{X_{c_\alpha}^L} \right) (\psi, a) \end{aligned}$$

■

Let (v_α) be an orthonormal basis in \mathfrak{s}^\perp , then $(\rho_\alpha) := (k(\Lambda_{v_\alpha}\mathfrak{s}))$ is an orthonormal basis in \mathfrak{a}^0 . Straightforward computations prove that elements $c_\alpha := -\Lambda_{v_\alpha}\mathfrak{f}$ satisfy condition stated in the lemma above.

Now, using (17) and the decomposition above, we have:

$$\begin{aligned} (20) \quad \{\tilde{k}_\varphi, \tilde{k}_\psi\} &= \sum_{\alpha\beta} \text{sgn}(\rho_\alpha) \text{sgn}(\rho_\beta) \{\tilde{k}_{\varphi\rho_\alpha} \widetilde{X_{c_\alpha}^L}, \tilde{k}_{\psi\rho_\beta} \widetilde{X_{c_\beta}^L}\} = \\ &= \sum_{\alpha\beta} \text{sgn}(\rho_\alpha) \text{sgn}(\rho_\beta) \left[\{\widetilde{X_{c_\alpha}^L}, \tilde{k}_{\psi\rho_\beta}\} \tilde{k}_{\varphi\rho_\alpha} \widetilde{X_{c_\beta}^L} - \{\widetilde{X_{c_\beta}^L}, \tilde{k}_{\varphi\rho_\alpha}\} \tilde{k}_{\psi\rho_\beta} \widetilde{X_{c_\alpha}^L} + \widetilde{X_{[c_\alpha, c_\beta]}^L} \tilde{k}_{\varphi\rho_\alpha} \tilde{k}_{\psi\rho_\beta} \right] = \\ &=: \boxed{\text{I}} + \boxed{\text{II}} + \boxed{\text{III}} \end{aligned}$$

To end our computations we need formula for $\Pi^L(X_p^L)(\tilde{k}_{\varphi\psi})$ for $p := \Lambda_{\mathbf{x}\mathbf{f}} \in \mathfrak{c}$, $x \in \mathfrak{s}^\perp$ (note that the same symbol $\tilde{k}_{\varphi\psi}$ is used for function on $\mathfrak{a}^0 \times A$ and on A). By (15) and (16):

$$(21) \quad \Pi^L(X_p^L)(a) = Za, \text{ where } Z := \text{ad}(a)\Lambda_{\mathbf{x}\mathbf{t}} - d(a)\Lambda_{(ax)\mathbf{t}}, a \in A, as =: d(a)\mathbf{s}$$

and

$$\begin{aligned} \Pi^L(X_p^L)(\tilde{k}_{\varphi\psi})(a) &= \frac{d}{dt} \Big|_{t=0} \tilde{k}_{\varphi\psi}(\exp(Zt)a) = \frac{d}{dt} \Big|_{t=0} \tilde{k}(\varphi, \text{ad}^\#(\exp(Zt)a)\psi) = \\ &= \frac{d}{dt} \Big|_{t=0} k(\Lambda_{v\mathbf{s}}, \text{ad}(\exp(Zt))\text{ad}(a)\Lambda_{w\mathbf{s}}) = \frac{d}{dt} \Big|_{t=0} k(\text{ad}(\exp(-Zt))\Lambda_{v\mathbf{s}}, \text{ad}(a)\Lambda_{w\mathbf{s}}), \end{aligned}$$

where we put $\varphi = k(\Lambda_{v\mathbf{s}}), \psi = k(\Lambda_{w\mathbf{s}})$ for $v, w \in \mathfrak{s}^\perp$. We have the equality $\text{ad}(\exp(-Zt))\Lambda_{v\mathbf{s}} = \Lambda(\exp(-Zt)v, \exp(-Zt)\mathbf{s})$, where for a while we use $\Lambda(x, y)$ for Λ_{xy} . Since we are ineterested only in derivative in $t = 0$ we can replace $\exp(-Zt)$ by $I - Zt$ and get:

$$(22) \quad \frac{d}{dt} \Big|_{t=0} k(\text{ad}(\exp(-Zt))\Lambda_{v\mathbf{s}}, \text{ad}(a)\Lambda_{w\mathbf{s}}) = k(-\Lambda_{(Zv)\mathbf{s}} - \Lambda_{v(Z\mathbf{s})}, \text{ad}(a)\Lambda_{w\mathbf{s}})$$

By (21) $Z\mathbf{s} = 0$ and :

$$Zv = [\Lambda_{(ax)(a\mathbf{t})} - d(a)\Lambda_{(ax)\mathbf{t}}]v = [\eta(a\mathbf{t}, v) - d(a)\eta(\mathbf{t}, v)](ax) - \eta(ax, v)(a\mathbf{t}) + d(a)\eta(ax, v)\mathbf{t}.$$

Therefore

$$\begin{aligned} \Lambda_{(Zv)\mathbf{s}} + \Lambda_{v(Z\mathbf{s})} &= \Lambda_{(Zv)\mathbf{s}} = [\eta(a\mathbf{t}, v) - d(a)\eta(\mathbf{t}, v)]\Lambda_{(ax)\mathbf{s}} - \eta(ax, v)\Lambda_{(a\mathbf{t})\mathbf{s}} + d(a)\eta(ax, v)\Lambda_{\mathbf{t}\mathbf{s}} \\ &= [\eta(a\mathbf{t}, v) - d(a)\eta(\mathbf{t}, v)]d(a)\Lambda_{(ax)(a\mathbf{s})} - \eta(ax, v)d(a)\Lambda_{(a\mathbf{t})(a\mathbf{s})} + d(a)\eta(ax, v)\Lambda_{\mathbf{t}\mathbf{s}} = \\ &= [\eta(a\mathbf{t}, v) - d(a)\eta(\mathbf{t}, v)]d(a)\text{ad}(a)\Lambda_{x\mathbf{s}} - \eta(ax, v)d(a)\text{ad}(a)\Lambda_{\mathbf{t}\mathbf{s}} + d(a)\eta(ax, v)\Lambda_{\mathbf{t}\mathbf{s}} \end{aligned}$$

So (22) is equal to

$$(23) \quad -[\eta(a\mathbf{t}, v) - d(a)\eta(\mathbf{t}, v)]d(a)k(\Lambda_{x\mathbf{s}}, \Lambda_{w\mathbf{s}}) + \eta(ax, v)d(a)k(\Lambda_{\mathbf{t}\mathbf{s}}, \Lambda_{w\mathbf{s}}) - d(a)\eta(ax, v)k(\Lambda_{\mathbf{t}\mathbf{s}}, \text{ad}(a)\Lambda_{w\mathbf{s}})$$

Let us define $\rho := k(\Lambda_{x\mathbf{s}})$ and $\boldsymbol{\rho} := k(\Lambda_{\mathbf{t}\mathbf{s}})$, then we have (recall that $\varphi = k(\Lambda_{v\mathbf{s}}), \psi = k(\Lambda_{w\mathbf{s}})$):

$$k(\Lambda_{x\mathbf{s}}, \Lambda_{w\mathbf{s}}) = \tilde{k}(\rho, \psi), \quad k(\Lambda_{\mathbf{t}\mathbf{s}}, \Lambda_{w\mathbf{s}}) = \tilde{k}(\boldsymbol{\rho}, \varphi), \quad k(\Lambda_{\mathbf{t}\mathbf{s}}, \text{ad}(a)\Lambda_{w\mathbf{s}}) = \tilde{k}(\boldsymbol{\rho}, \text{ad}^\#(a)\psi) = \tilde{k}_{\boldsymbol{\rho}\psi}(a),$$

$$d(a)\eta(ax, v) = d(a)k(\Lambda_{(ax)\mathbf{s}}, \Lambda_{v\mathbf{s}}) = k(\text{ad}(a)\Lambda_{x\mathbf{s}}, \Lambda_{v\mathbf{s}}) = \tilde{k}(\text{ad}^\#(a)\rho, \varphi) = \tilde{k}_{\varphi\rho}(a)$$

$$d(a)\eta(a\mathbf{t}, v) = d(a)k(\Lambda_{(a\mathbf{t})\mathbf{s}}, \Lambda_{v\mathbf{s}}) = \tilde{k}_{\varphi\rho}(a), \quad \eta(\mathbf{t}, v) = \tilde{k}(\boldsymbol{\rho}, \varphi)$$

and (23) is equal to:

$$\begin{aligned} & \left[\tilde{k}(\boldsymbol{\rho}, \varphi) - \tilde{k}_{\varphi\rho}(a) \right] \tilde{k}(\rho, \psi) + \tilde{k}_{\varphi\rho}(a) \left[\tilde{k}(\boldsymbol{\rho}, \varphi) - \tilde{k}_{\boldsymbol{\rho}\psi}(a) \right] = \\ & = \left\{ \tilde{k}(\rho, \psi) \left[\tilde{k}(\boldsymbol{\rho}, \varphi)I - \tilde{k}_{\varphi\rho} \right] + \tilde{k}_{\varphi\rho} \left[\tilde{k}(\boldsymbol{\rho}, \varphi)I - \tilde{k}_{\boldsymbol{\rho}\psi} \right] \right\} (a) \end{aligned}$$

In this way we finally get:

$$(24) \quad \Pi^L(X_p^L)(\tilde{k}_{\varphi\psi}) = \tilde{k}(\rho, \psi) \left[\tilde{k}(\boldsymbol{\rho}, \varphi)I - \tilde{k}_{\varphi\rho} \right] + \tilde{k}_{\varphi\rho} \left[\tilde{k}(\boldsymbol{\rho}, \varphi)I - \tilde{k}_{\boldsymbol{\rho}\psi} \right],$$

where $p := \Lambda_{\mathbf{x}\mathbf{f}}$, $x \in \mathfrak{s}^\perp$, $\rho := k(\Lambda_{x\mathbf{s}})$ and $\boldsymbol{\rho} := k(\Lambda_{\mathbf{t}\mathbf{s}})$.

Now we return to computations of (20). Choose an orthonormal basis (e_α) in \mathfrak{s}^\perp with $e_0 := \mathbf{t}$. Then we have orthonormal basis $\rho_\alpha := k(\Lambda_{e_\alpha\mathbf{s}}) =: k(\Lambda_{\alpha\mathbf{s}})$ in \mathfrak{a}^0 with $\rho_0 = \boldsymbol{\rho}$ and corresponding elements $c_\alpha := -\Lambda_{e_\alpha\mathbf{f}} =: -\Lambda_{\alpha\mathbf{f}}$. Using (24) we obtain:

$$\begin{aligned} \{\widetilde{X_{c_\alpha}^L}, \tilde{k}_{\psi\rho_\beta}\} &= \Pi^L(X_{c_\alpha}^L)(\tilde{k}_{\psi\rho_\beta}) = \tilde{k}(-k(\Lambda_{\alpha\mathbf{s}}), k(\Lambda_{\beta\mathbf{s}})) \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\psi\rho} \right] - \tilde{k}_{\psi\rho_\alpha} \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\boldsymbol{\rho}\rho_\beta} \right] = \\ &= -\tilde{k}(\rho_\alpha, \rho_\beta) \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\psi\rho} \right] - \tilde{k}_{\psi\rho_\alpha} \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\boldsymbol{\rho}\rho_\beta} \right] = \\ &= -\text{sgn}(\rho_\alpha)\delta_{\alpha\beta} \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\psi\rho} \right] - \tilde{k}_{\psi\rho_\alpha} \left[\tilde{k}(\boldsymbol{\rho}, \psi)I - \tilde{k}_{\boldsymbol{\rho}\rho_\beta} \right] \end{aligned}$$

In this way the first term in the sum (20) is equal to:

$$\begin{aligned} \boxed{\text{I}} &= \sum_{\alpha\beta} \text{sgn}(\rho_\alpha) \text{sgn}(\rho_\beta) \{ \widetilde{X_{c_\alpha}^L}, \tilde{k}_{\psi\rho_\beta} \} \tilde{k}_{\varphi\rho_\alpha} \widetilde{X_{c_\beta}^L} = -\tilde{k}_\varphi \left(\tilde{k}(\boldsymbol{\rho}, \psi) I - \tilde{k}_{\psi\rho} \right) - \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha} \tilde{k}_{\psi\rho_\alpha} \widetilde{X_{c_0}^L} + \\ &\quad + \tilde{k}_\rho \sum_{\alpha} \text{sgn}(\rho_\alpha) \tilde{k}_{\varphi\rho_\alpha} \tilde{k}_{\psi\rho_\alpha} \end{aligned}$$

The second term in (20) we get by interchanging in $\boxed{\text{I}}$ α with β , φ with ψ and changing the sign:

$$\boxed{\text{II}} = \tilde{k}_\psi \left(\tilde{k}(\boldsymbol{\rho}, \varphi) I - \tilde{k}_{\varphi\rho} \right) + \sum_{\beta} \text{sgn}(\rho_\beta) \tilde{k}_{\psi\rho_\beta} \tilde{k}_{\varphi\rho_\beta} \widetilde{X_{c_0}^L} - \tilde{k}_\rho \sum_{\beta} \text{sgn}(\rho_\beta) \tilde{k}_{\psi\rho_\beta} \tilde{k}_{\varphi\rho_\beta}$$

and their sum is

$$(25) \quad \boxed{\text{I}} + \boxed{\text{II}} = \tilde{k}_\psi \left(\tilde{k}(\boldsymbol{\rho}, \varphi) I - \tilde{k}_{\varphi\rho} \right) - \tilde{k}_\varphi \left(\tilde{k}(\boldsymbol{\rho}, \psi) I - \tilde{k}_{\psi\rho} \right)$$

It remains to compute $\boxed{\text{III}}$.

$$[c_\alpha, c_\beta] = [-\Lambda_{\alpha\mathbf{f}}, -\Lambda_{\beta\mathbf{f}}] = \eta(f, v_\beta) \Lambda_{\alpha\mathbf{f}} - \eta(f, v_\alpha) \Lambda_{\beta\mathbf{f}} = \delta_{0\alpha} c_\beta - \delta_{0\beta} c_\alpha$$

therefore

$$\widetilde{X_{[c_\alpha, c_\beta]}^L} = \delta_{0\alpha} \widetilde{X_{c_\beta}^L} - \delta_{0\beta} \widetilde{X_{c_\alpha}^L}$$

and

$$\boxed{\text{III}} = \sum_{\alpha\beta} \text{sgn}(\rho_\alpha) \text{sgn}(\rho_\beta) \tilde{k}_{\varphi\rho_\alpha} \tilde{k}_{\psi\rho_\beta} \left[\delta_{0\alpha} \widetilde{X_{c_\beta}^L} - \delta_{0\beta} \widetilde{X_{c_\alpha}^L} \right] = \tilde{k}_\psi \tilde{k}_{\varphi\rho} - \tilde{k}_\varphi \tilde{k}_{\psi\rho}$$

Finally:

$$(26) \quad \{ \tilde{k}_\varphi, \tilde{k}_\psi \} = \boxed{\text{I}} + \boxed{\text{II}} + \boxed{\text{III}} = \tilde{k}(\boldsymbol{\rho}, \varphi) \tilde{k}_\psi - \tilde{k}(\boldsymbol{\rho}, \psi) \tilde{k}_\varphi$$

In the similar way, using lemma 2.1 and formulae (19) and (24) we obtain:

$$\{ \tilde{k}_\lambda, \tilde{k}_{\varphi\psi} \} = \tilde{k}_{\lambda\psi} (\tilde{k}_{\varphi\rho} - \tilde{k}(\boldsymbol{\rho}, \varphi) I) + \tilde{k}(\lambda, \varphi) (\tilde{k}_{\rho\psi} - \tilde{k}(\boldsymbol{\rho}, \psi) I)$$

Now we have all the brackets:

$$(27) \quad \begin{aligned} \{ \tilde{k}_\varphi, \tilde{k}_\psi \} &= \tilde{k}(\boldsymbol{\rho}, \varphi) \tilde{k}_\psi - \tilde{k}(\boldsymbol{\rho}, \psi) \tilde{k}_\varphi, \\ \{ \tilde{k}_\lambda, \tilde{k}_{\varphi\psi} \} &= \tilde{k}_{\lambda\psi} (\tilde{k}_{\varphi\rho} - \tilde{k}(\boldsymbol{\rho}, \varphi) I) + \tilde{k}(\lambda, \varphi) (\tilde{k}_{\rho\psi} - \tilde{k}(\boldsymbol{\rho}, \psi) I), \\ \{ \tilde{k}_{\varphi\lambda}, \tilde{k}_{\psi\rho} \} &= 0 \text{ for } \varphi, \lambda, \psi, \rho \in \mathfrak{a}^0 \text{ and } \boldsymbol{\rho} := k(\Lambda_{\mathbf{ts}}). \end{aligned}$$

The Poincaré group in [4] was identified with matrices $g = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix}$, where Λ is a Lorentz matrix of dimension $n+1$ and $v \in \mathbb{R}^{n+1}$. Poisson brackets for matrix elements of g are given by:

$$(28) \quad \begin{aligned} \{ \Lambda_{\mu\nu}, v_\beta \} &= h [(\Lambda_{\mu 0} - \delta_{\mu 0}) \Lambda_{\beta\nu} + \eta_{\mu\beta} (\Lambda_{0\nu} - \delta_{0\nu})], \\ \{ v_\alpha, v_\beta \} &= h (v_\alpha \delta_{\beta 0} - v_\beta \delta_{\alpha 0}), \\ \{ \Lambda_{\mu\nu}, \Lambda_{\alpha\beta} \} &= 0, \end{aligned}$$

where $\eta_{\alpha\beta} := \text{diag}(1, -1, \dots, -1)$ and h is a real parameter (**Note:** here $\Lambda_{\alpha\beta}$ are *matrix elements not operators*). To compare the brackets, let us choose an orthonormal basis $(\rho_\alpha) \in \mathfrak{a}^0$ with $\rho_0 = \boldsymbol{\rho}$. We have

$$\tilde{k}(\rho_\alpha, \rho_\beta) = \text{diag}(1, -1, \dots, -1) = \eta_{\alpha\beta}, \quad v_\alpha = \text{sgn}(\rho_\alpha) \tilde{k}_{\rho_\alpha} \text{ and } \Lambda_{\alpha\beta} = \text{sgn}(\rho_\alpha) \tilde{k}_{\rho_\alpha \rho_\beta}.$$

Short computations show that brackets (27) coincide with (28) for $h = -1$.

3. POISSON MINKOWSKI SPACE

Let (V, η) be a real, n -dimensional ($n > 2$) vector space with a symmetric, bilinear, nondegenerate form η . For a basis (v_α) of V let $\eta_{\alpha\beta} := \eta(v_\alpha, v_\beta)$ be the corresponding matrix of η and $\eta^{\alpha\beta}$ stands for the inverse matrix. *Note that despite the title of the section, (V, η) needn't to be a (vector) Minkowski space.* In this section G denotes any subgroup of $O(\eta)$ containing $SO_0(\eta)$ and $IG := V \rtimes G$ is the semi-direct product. The Lie algebra of IG is $iso(\eta) := V \times so(\eta)$ and the bracket is:

$$(29) \quad [(v, A), (w, B)] = (Aw - Bv, [A, B])$$

The Poisson bracket for κ -Poincaré in [4] is an example of a more general situation [6]. For a vector $v \in V$ let us define

$$(30) \quad b_v := \sum \eta^{jk} e_j \wedge \Lambda_{v, e_k} \in iso(\eta) \wedge iso(\eta),$$

where (e_k) is any basis in V . Direct computation proves that, for $u, v \in V$, elements b_v, b_u satisfy:

$$(31) \quad [b_v, b_u] = -\eta(v, u)\Omega,$$

where $\Omega := \sum \eta^{jk} \eta^{mn} e_j \wedge e_m \wedge \Lambda_{e_k, e_n}$ is the canonical invariant element in $iso(\eta) \wedge iso(\eta) \wedge iso(\eta)$, and

$$(32) \quad [a \wedge b, c \wedge d] := a \wedge [b, c] \wedge d - a \wedge [b, d] \wedge c - b \wedge [a, c] \wedge d + b \wedge [a, d] \wedge c$$

is the (algebraic) Schouten bracket. Therefore b_v defines a Poisson-Lie structure $\hat{\Pi}_v$ on IG by:

$$(33) \quad \hat{\Pi}_v(g) = b_v g - g b_v$$

The structure in [4] is of this type for v being a timelike vector. Moreover, it is easy to see that

$$(34) \quad [b_v, x \wedge u] = 2u \wedge x \wedge v, \text{ for any } x, u \in V$$

so we can replace b_v in (33) by $b_v + x \wedge v$ and we obtain another Poisson-Lie structure on IG which will be denoted by $\hat{\Pi}_{v,x}$. The adjoint representation of IG on $iso(\eta)$ is given by:

$$\text{ad}_{(w,A)}(v, X) = (w + Av - AXA^{-1}w, AXA^{-1}), \quad w, v \in V, \quad A \in O(\eta), \quad X \in so(\eta);$$

by the same symbol we will denote this representation canonically extended to $iso(\eta) \wedge iso(\eta)$. Straightforward computations give:

$$(35) \quad \text{ad}_{(w,A)}(b_v) = w \wedge Av + b_{Av}, \quad \text{ad}_{(w,A)}(x \wedge v) = Ax \wedge Av$$

Let (M, V, η) be an affine space modeled on (V, η) . Let $Aff(G)$ be the group of those affine isometries of M that have G as their linear part. Any point $m \in M$ defines the isomorphism $\phi_m : IG \rightarrow Aff(G)$ given by:

$$\phi_m(w, A)(m + v) := m + w + Av, \quad v \in V$$

For two points $m, n \in M$ we have: $\phi_m^{-1} \phi_n = Ad_{n-m} : IG \ni g \mapsto (n - m)g(n - m)^{-1} \in IG$ – the inner automorphism given by $n - m \in V$. In this way for a point $m \in M$ and a vector $v \in V$ we have the Poisson-Lie structure on $Aff(G)$ defined by:

$$(36) \quad \Pi_{m,v} := \phi_m(\hat{\Pi}_v)$$

Proposition 3.1. *Let $\Pi_{m,v}$ be the Poisson structure defined in (36). Then:*

- $\Pi_{m,\lambda v} = \lambda \Pi_{m,v}$, $\Pi_{m+\lambda v, v} = \Pi_{m,v}$ i.e. the bivector $\Pi_{m,v}$ depends only on a parametrized line $l := \{m + tv, t \in \mathbb{R}\}$; we will write Π_l for this Poisson structure.
- Let l, k be two parametrized lines then $\Pi_l = \Pi_k$ iff $l = k$.
- If $\dim(V) > 3$ then Π_l and Π_k are compatible iff l and k intersect or are parallel; for $\dim(V) = 3$: if $G \subset SO(\eta)$ then any two structures Π_l and Π_k are compatible; otherwise the statement is as for $\dim(V) > 3$.

Proof: The equality $\Pi_{m,\lambda v} = \lambda \Pi_{m,v}$ is obvious. Let $m, n \in M$, $v, u \in V$ and $x := n - m \in V$. We can transfer $\Pi_{n,u}$ to IG by ϕ_m^{-1} and get $\phi_m^{-1} \phi_n(b_u) = \text{ad}_x(b_u) = x \wedge u + b_u$ by (35). Taking $n := m + \lambda v$ we get the second equality.

Let lines l, k be given by (m, v) and (n, u) respectively. Then $l \neq k$ means that $v \neq u$ or if $v = u$ then $x := n - m \neq 0$ and x, v are linearly independent. Using the definition (30) and the formula above it is easy to prove the second statement.

Poisson structures Π_l and Π_k are compatible iff $\hat{\Pi}_v + \hat{\Pi}_{u,x}$ ($x := n - m$) is a Poisson structure on IG , i.e. the Schouten bracket $[\hat{\Pi}_v + \hat{\Pi}_{u,x}, \hat{\Pi}_v + \hat{\Pi}_{u,x}] = 0$ and (since $\hat{\Pi}_v$ and $\hat{\Pi}_{u,x}$ are Poisson) this is equivalent to $[\hat{\Pi}_v, \hat{\Pi}_{u,x}] = 0$. By (33) this, in turn, is equivalent to $[b_v, x \wedge u + b_u]$ being IG invariant (with respect to adjoint action). Using (31) and (34) we get that $x \wedge u \wedge v$ must be G invariant. Clearly this element is 0 for intersecting or parallel lines l and k . For $\dim(V) > 3$, G , invariance of $x \wedge u \wedge v$ forces it to be 0, i.e. lines l and k intersect or are parallel; for $\dim(V) = 3$ the element $x \wedge u \wedge v$ is invariant if G preserves orientation. \blacksquare

A parametrized line $l := \{m + tw, t \in \mathbb{R}\}$ defines also a bivector π_l on M :

$$(37) \quad \pi_l(m + v) := v \wedge w,$$

in the formula above we identify TM with $M \times V$; it is easy to see that really π_l depends only on l and not on the chosen point $m \in l$.

Proposition 3.2. • π_l is a Poisson bivector on M .

- π_l and π_k are compatible iff lines l, k intersect or are parallel.
- The canonical action of $(\text{Aff}(G), \Pi_k)$ on (M, π_l) is Poisson iff $l = k$.

Proof: Let $l := \{m + tw, t \in \mathbb{R}\}$ and define the vector field \hat{V}^m by $\hat{V}^m(m + v) := v$; let \hat{w} be the constant vector field: $\hat{w}(m + v) := w$; with this notation we have: $\pi_l = \hat{V}^m \wedge \hat{w}$. If π_k is defined by the line $k := \{n + tu, t \in \mathbb{R}\}$ then

$$\pi_k(m + v) = \pi_k(n + (m - n) + v) = (m - n) \wedge u + v \wedge u = (\hat{x} \wedge \hat{u} + \hat{V}^m \wedge \hat{u})(m + v),$$

where $x := m - n$, i.e. $\pi_k = \hat{x} \wedge \hat{u} + \hat{V}^m \wedge \hat{u}$. Let us compute:

$$\begin{aligned} [\pi_l, \pi_k] &= [\hat{V}^m \wedge \hat{w}, \hat{x} \wedge \hat{u} + \hat{V}^m \wedge \hat{u}] = [\hat{V}^m \wedge \hat{w}, \hat{x} \wedge \hat{u}] + [\hat{V}^m \wedge \hat{w}, \hat{V}^m \wedge \hat{u}] = \\ &= -\hat{w} \wedge [\hat{V}^m, \hat{x}] \wedge \hat{u} + \hat{w} \wedge [\hat{V}^m, \hat{u}] \wedge \hat{x} + \hat{V}^m \wedge [\hat{w}, \hat{V}^m] \wedge \hat{u} + \hat{w} \wedge [\hat{V}^m, \hat{u}] \wedge \hat{V}^m \end{aligned}$$

But for any constant vector field \hat{y} we have: $[\hat{V}^m, \hat{y}] = -\hat{y}$, therefore:

$$[\pi_l, \pi_k] = 2\hat{w} \wedge \hat{x} \wedge \hat{u}.$$

In this way $[\pi_l, \pi_l] = 0$ and $\pi_l + \pi_k$ is Poisson iff $w \wedge x \wedge u = 0$. Now first and the second statement are clear.

Let the lines l, k be defined by (m, w) and (n, u) , respectively; let $\psi_n : V \ni v \mapsto n + v \in M$. Using ϕ_n and ψ_n we can transfer problem to the action on $(IG, \hat{\Pi}_u)$ on $(V, \hat{\pi}_l)$, where $\hat{\Pi}_u$ is defined by (33) and $\psi_n(\hat{\pi}_l) = \pi_l$ i.e. $\hat{\pi}_l(v) = (x + v) \wedge w$, $x := n - m$. The action is

$$IG \times V \ni (y, A; v) \mapsto y + Av \in V$$

This action is Poisson iff

$$(38) \quad \hat{\pi}_l(gv) = \hat{g}\hat{\pi}_l(v) + \hat{\Pi}_u(g)\hat{v}, \quad g := (y, A) \in IG,$$

where \hat{g} is (the extension of) the mapping $V \ni v \mapsto gv \in V$ and \hat{v} (the extension of) $IG \ni g \mapsto gv \in V$. We have:

$$\begin{aligned} \hat{\pi}_l(gv) &= \hat{\pi}_l(y + Av) = (x + y + Av) \wedge w \\ \hat{g}(\hat{\pi}_l(v)) &= (Ax + Av) \wedge Aw \\ \hat{\Pi}_u(g)\hat{v} &= (b_u g - g b_u)\hat{v} = (b_u)\widehat{g}\hat{v} - \hat{g}(b_u\hat{v}) \end{aligned}$$

It is straightforward, that for $(\dot{x}, \dot{A}) \in T_e IG : (\dot{x}, \dot{A})\hat{z} = \dot{x} + \dot{A}z$; so

$$(b_u)\hat{z} = \sum \eta^{jk} e_j \wedge (\Lambda_{ue_k} z) = \sum \eta^{jk} e_j \wedge (\eta(e_k, z)u - \eta(u, z)e_k) = z \wedge u$$

therefore

$$(b_u)\widehat{g\hat{v}} = (y + Av) \wedge u, \quad (b_u)\hat{v} = v \wedge u$$

and

$$\hat{g}(b_u \hat{v}) = \hat{g}(v \wedge u) = Av \wedge Au$$

In this way equality (38) reads:

$$(x + y + Av) \wedge w = (Ax + Av) \wedge Aw + (y + Av) \wedge u - Av \wedge Au, \text{ for any } y, v \in V, A \in G$$

If $l = k$ i.e. $x = 0, w = u$ this condition is fulfilled. On the other hand, setting $v = 0, A = I$ we get (for any y) $y \wedge w = y \wedge u$, so $w = u$ and the equality reduces to

$$x \wedge w = Ax \wedge Aw \text{ for any } A \in G$$

Therefore $x \wedge w = 0$ and $l = k$. ■

REFERENCES

- [1] J Lukierski, A Nowicki, H Ruegg, *New quantum Poincaré algebra and -deformed field theory* Physics Letters B, **293** (1992), pp 344-352.
- [2] S Majid, H Ruegg, *Bicrossproduct structure of -Poincaré group and non-commutative geometry*, Physics Letters B, **334** (1994), pp 348-354.
- [3] S. Vaes, L. Vainerman, *Extensions of locally compact quantum groups and the bicrossed product construction*, Advances in Mathematics **175** (1) (2003), 1–101 .
- [4] S. Zakrzewski *Quantum Poincaré group related to the κ -Poincaré algebra*. J. Phys. A Math. Gen. **27** (1994) 2075-2082.
- [5] P. Stachura *On the quantum $ax + b$ group*, J. Geom. Phys. **73** (2013), 125-149.
- [6] S. Zakrzewski *Poisson Structures on the Poincaré Group*, Comm. in Math. Phys. **185** (1997), pp 285-311.
- [7] S. Zakrzewski, *Poisson homogeneous spaces*, in: J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Quantum groups (Karpacz, 1994), PWN, Warszawa, 1995, pp 629-639.
- [8] S. Zakrzewski, *Quantum and classical pseudogroups. II. Differential and symplectic pseudogroups*, Comm. Math. Phys. **134**, (1990), pp 71-395.

FACULTY OF APPLIED INFORMATICS AND MATHEMATICS, WARSAW UNIVERSITY OF LIFE SCIENCES-SGGW, UL NOWOURSYNOWSKA 166, 02-787 WARSZAWA, POLAND, E-MAIL: STACHURA@FUW.EDU.PL